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A METHOD OF SOLUTION OF THE HEAT CONDUCTION (DIFFUSION) EQUATION

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Аннотация—Разработан метод решения уравнения теплопроводности (диффузии) для нестационарной трехмерной задачи при наличии конвекции. Дано теоретическое обоснование метода для случая, когда горизонтальная (продольная) составляющая скорости перемещения среды суть произвольный полином относительно вертикальной координаты. Решение уравнения базируется на операционном методе и имеет вид ряда, расположенного по степеням обобщенного параметра Лапласа.

NOMENCLATURE

T, temperature;
x, y, horizontal plane;
z, vertical co-ordinate;
$u(z) = \sum_{m=0} a_m z^m$ and
W, corresponding velocities of medium displacement;
λ, λ_z, corresponding coefficients of heat conduction;
T_0, Green's function.

1. METHOD OF SOLVING EQUATION (1.1)

The heat equation:

$$\frac{\partial T}{\partial t} + W\frac{\partial T}{\partial z} + u(z)\frac{\partial T}{\partial x} = \lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \lambda_z\frac{\partial^2 T}{\partial z^2} \quad (1.1)$$

is solved for the following initial and boundary conditions, assuming that W, λ and λ_z are constant:

at

$$t = 0, \quad T = Q\delta(x)\,\delta(y)\,\delta(z-h);$$

at

$$\sqrt{(x^2+y^2+z^2)} \to \infty \quad T = 0;$$

$$\left(\lambda_z\frac{\partial T}{\partial z} + WT\right)_{z=0} = 0.$$

The method of solution is illustrated in application to the equation:

$$\frac{\partial T}{\partial t} + W\frac{\partial T}{\partial z} + az\frac{\partial T}{\partial x} = \lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \lambda_z\frac{\partial^2 T}{\partial z^2}. \quad (1.2)$$

Equation (1.2) is solved instead of (1.1) only to simplify the presentation, and to avoid cumbersome computations which hinder an understanding of the method.

In future it will not be essential that a horizontal velocity component of medium displacement be given as a linear function of the co-ordinate z. The method proposed allows for solution when the horizontal velocity component of medium displacement is given as polynomials of any whole powers with respect to z.

Substituting the variables in equation (1.2):

$$\xi = \frac{x}{\sqrt{\lambda}},\ \eta = \frac{y}{\sqrt{\lambda}},\ \zeta = \frac{z}{\sqrt{\lambda_z}},\ W = \frac{W}{\sqrt{\lambda_t}},$$

we obtain:

$$\frac{\partial T}{\partial t} + W\frac{\partial T}{\partial \zeta} + b\zeta\frac{\partial T}{\partial \xi} = \frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2} + \frac{\partial^2 T}{\partial \zeta^2}, \quad (1.3)$$

where

$$b = a\sqrt{\left(\frac{\lambda_z}{\lambda}\right)}.$$

Equation (1.3) is solved by the operational method. With the help of the Laplace triple transformation:

$$T^* = \int_0^\infty T\, e^{-P_t t}\, dt \iint_{-\infty}^{\infty} e^{-P_\xi \xi - P_\eta \eta}\, d\xi\, d\eta$$

substituting

$$T^* = \chi\, e^{(w/2)\zeta}$$

equation (1.3) is reduced to the form:

$$\frac{d^2\chi}{d\zeta^2} - P\chi = bP_\xi \zeta\chi - \frac{Q}{\lambda} e^{-(W/2)\zeta}\, \delta[\zeta\sqrt{(\lambda_t)} - h], \qquad (1.4)$$

where

$$P = P_t - P_\xi^2 - P_\eta^2 - \frac{W^2}{4}.$$

Assuming that $\zeta\sqrt{(P)} = \zeta^*$, we reduce equation (1.4) to the form:

$$\frac{d^2\chi}{d\zeta^{*2}} - [1 + 2\zeta^*]\chi = \frac{Q}{P\lambda}\, e^{-W\zeta^*/2\sqrt{(P)}}\, \delta\left(\frac{\lambda_z}{P}\zeta^* - h\right), \qquad (1.5)$$

where

$$\frac{bP_\xi}{P^{3/2}}.$$

Let us solve the homogeneous equation:

$$\frac{d^2\chi}{d\zeta^{*2}} - [1 + \alpha\zeta^*]\chi = 0. \qquad (1.6)$$

We shall find the solution of this equation in the form of the following series arranged according to the powers of α:

$$\chi = \chi_0 + \alpha\chi_1 + \alpha^2\chi_2 + \ldots + \alpha^n\chi_n + \ldots \qquad (1.7)$$

Substituting equation (1.7) into (1.6) and equating the terms of equal powers of α, we get the following infinite system of ordinary differential equations:

$$\left.\begin{aligned} &\frac{d^2\chi_0}{d\zeta^{*2}} - \chi_0 = 0,\\ &\frac{d^2\chi_1}{d\zeta^{*2}} - \chi_1 = \zeta^*\chi_0,\\ &\frac{d^2\chi_n}{d\zeta^{*2}} - \chi_n = \zeta^*\chi_{n-1}.\end{aligned}\right\} \qquad (1.8)$$

The first equation of system (1.8) has two fundamental solutions:

$$e^{\zeta^*} \text{ and } e^{-\zeta^*}.$$

The two fundamental solutions for equation (1.5) may be obtained with the help of the generating operator:

$$\chi_n = \tfrac{1}{2}\,[e^{\zeta^*}\int e^{-\zeta^*}\varphi_{n-1}(\zeta^*)\, d\zeta^* - e^{-\zeta^*}\int e^{\zeta^*}\varphi_{n-1}(\zeta^*) d\zeta^*], \qquad (1.9)$$

where

$$\varphi_{n-1}(\zeta^*) = \zeta^*\chi_{n-1}(\zeta^*).$$

Assuming first $\chi_0 = e^{\zeta^*}$ we get the first fundamental solution of equation (1.6) in the form: $\chi_1 = e^{\zeta^*}\mathscr{F}_1(\zeta^*)$. The second fundamental solution of (1.6) may be obtained in an analogous way:

$$\chi_2 = e^{-\zeta^*}\mathscr{F}_2(\zeta^*).$$

The general solution of equation (1.5) for the boundary conditions given above will be written as follows [1]:

$$\left.\begin{aligned} \chi = {}& \frac{Q\, e^{-(Wh/2\lambda_z)}}{P\lambda\sqrt{\lambda_z}}\, \frac{1}{W}\Bigg[\chi_1(h, P) + \frac{(W/2)\chi_1(0) - \sqrt{(P\lambda_z)}\,\chi_1'(0)}{-(W/2)\chi_2(0) + \sqrt{(P\lambda_z)}\,\chi_2'(0)}\chi_2(h, P)\Bigg]\\ &\times \chi_2(\zeta^*, P), \quad \zeta^* \geqslant h\sqrt{\left(\frac{P}{k_z}\right)}\ (z \geqslant h);\end{aligned}\right\} (1.10)$$

$$\left.\begin{aligned} \chi = {}& \frac{Q\, e^{-(Wh/2\lambda_t)}}{P\lambda\sqrt{\lambda_t}}\, \frac{1}{W}\Bigg[\chi_1(\zeta^*, P) + \frac{(W/2)\chi_1(0) - \sqrt{(P\lambda_z)}\,\chi_1'(0)}{-(W/2)\chi_2(0) + \sqrt{(P\lambda_z)}\,\chi_2'(0)}\\ &\times \chi_2(\zeta^*, P)\Bigg]\chi_2(h, P),\\ &\zeta^* \leqslant h\sqrt{\left(\frac{P}{k_z}\right)}\ (z \leqslant h);\end{aligned}\right\} (1.11)$$

where W is the Wronsky determinant.

$$\begin{vmatrix} \chi_1 & \chi_2 \\ \chi_1' & \chi_2' \end{vmatrix}$$

Expressions (1.10) and (1.11) represent the image of the function T.

The transition from the image to the original is carried out with the help of the formulae:

$$L^{-1}\left(\frac{1}{\lambda^{(n/2)-1}}\,e^{-a\sqrt{(\lambda)}}\right) = (4t)^{n/2}\, i^n \operatorname{erf} c\,\frac{a}{2\sqrt{t}} \quad (1.12)$$

and

$$L^{-1}_{\xi,\eta}\,\{P^n_\xi \exp[-(P^2_\xi+P^2_\eta)t]\} = \frac{(-1)^n}{4\pi t}\left(\frac{1}{2\sqrt{t}}\right)^n \exp[-(y^2/4\lambda t)-\omega^2) \quad (1.13)$$

Here $i^n \operatorname{erf} c\, x$ is the integral function of errors of the n-th order; H_n are Hermite's polynomials [2, 3]; $\omega = \xi/2\sqrt{t}$.

The solution for equation (1.13) has the following form:

$$T = \mathscr{F}\left[i^n \operatorname{erf} c\frac{z}{2\sqrt{(\lambda_z t)}},\ H_n\left(\frac{x}{2\sqrt{(k_x t)}}\right) y, z, t\right]. \quad (1.14)$$

Solution (1.14) obtained for an instantaneous point source is Green's function for equation (1.3), with more general assumptions on the time and space characteristics of an admixture source.

With the solution of equation (1.3) we may obtain the solution of any non-homogeneous problem corresponding to this equation. Indeed, letting equation (1.3) have the term $f(x, y, z, t)$ in the right-hand side, the solution satisfying non-homogeneous initial and boundary conditions:

$$T(x, y, z, 0) = \varphi(x, y, z),$$

$$\left[\lambda_z\frac{\partial T}{\partial z} + WT\right]_{z=0} = \psi(x, y, t),$$

at

$$\sqrt{(x^2+y^2+t^2)} \to \infty \quad T = 0,$$

will have the form:

$$\left.\begin{aligned} T(x, y, z, t) &= \frac{1}{Q}\int_0^t d\tau \iint_{-\infty}^{\infty} d\xi\, d\eta \\ &\times \int_0^{\infty} f(x-\xi, y-\eta, \zeta, t-\tau)T_0 \\ &\times (\xi, \eta, z, \tau, \zeta)\, d\zeta + \frac{1}{Q}\iint_{-\infty}^{\infty} d\xi\, d\eta \\ &\times \int^{\infty} \phi(x-\xi, y-\eta, \zeta)T_0 \\ &\times (\xi, \eta, z, t, \zeta)\, d\zeta - \frac{1}{Q}\int_0^t d\tau \\ &\times \iint_{-\infty}^{\infty} \psi(x-\xi, y-\eta, t-\tau)T_0 \\ &\times (\xi, \eta, z, \tau; 0)\, d\zeta\, d\eta). \end{aligned}\right\} \quad (1.15)$$

At the boundary condition:

$$T|_{z=0} = \psi_1(x, y, t)$$

the last term in (1.15) should be substituted by the expression

$$\frac{1}{Q}\int_0^t d\tau \iint_{-\infty}^{\infty} \psi_1(x-\xi, y-\eta, t-\tau) \times\left[\lambda_z\frac{\partial T_0(\xi, \eta, z, t; \zeta)}{\partial\zeta}\right]_{\zeta=0} d\xi\, d\eta.$$

Let a stationary source with intensity q of heat units per time unit be concentrated at the point $x = y = 0$, $z = h$. The temperature distribution may be obtained as a limit of solution for equation (1.3) at $t \to \infty$, with the right-hand side

$$f(x, y, z, t) = q\delta(x)\,\delta(y)\,\delta(z-h)$$

at homogeneous initial and boundary conditions.

The first term of equation (1.15) shows that

$$T_1(x, y, z; h) = \frac{q}{Q}\int_0^{\infty} T_0(x, y, z, t; h)\, dt.$$

2. THEORETICAL SUBSTANTIATION OF THE METHOD

To substantiate the method proposed above, the following theorems were proved:

(1) Let $\prod\limits_{k=0}^{n} a_k \sim n^{a[4]}$ (*) at sufficiently large n and $a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n \ldots$ or $a_0 > a_1 \ldots > a_n > \ldots$. Then, at sufficiently large n,

$$\prod_{k=0}^{n} a_{km} \sim n^{a/m}\, u.$$

Therefore:

$$\prod_{m=2}^{n}\left(1+\frac{1}{(m+1)k+m-1}\right)\sim n^{1/k+1}. \quad (2.1)$$

We used the symbol $\sim$ with the meaning $a_n \sim n^{\alpha}$ when $n \to \infty$, if the condition $a_n = 0(n^{\beta})$ is fulfilled when $\beta \leqslant \alpha$.

(2) Let the function $\mathscr{F}_1(\xi, z, t)$ be the original of the image $P_{\xi}^{n} f(P, z)$, and the function $\mathscr{F}_2(\xi, z, t)$, that of $P_{\xi}^{n+1} f(P, z)$. Then for a sufficiently large number n, excluding the neighbourhood of zeros of the function $\mathscr{F}_2(\xi, z, t)$,

$$\left|\frac{\mathscr{F}_2(\xi, z, t)}{\mathscr{F}_1(\xi, z, t)}\right| \sim \sqrt{\frac{n}{t}}. \quad (2.2)$$

(3)

$$\max \frac{i^n \operatorname{erf} c\, x}{i^{n-1} \operatorname{erf} c\, x} = \frac{i^n \operatorname{erf} c\, 0}{i^{n-1} \operatorname{erf} c\, 0} \sim \frac{1}{\sqrt{n}}. \quad (2.3)$$

The main idea of the proof that there exists the original for the general solution of the equation when $u(z) = \sum_{\nu=0}^{k} r_\nu \zeta^\nu$ is the following [1]:

We first prove the existence of the original for the function χ which is the solution for the equation:

$$\frac{d^2\chi}{d\zeta^2} - P\chi = P_{\xi} u(\zeta)\chi$$

or

$$\frac{d^2\chi}{d\zeta^{*2}} - \chi = \frac{1}{P} P_{\xi} u(\zeta^*)\chi. \quad (2.4)$$

The formal solution of equation (2.4) may be obtained in the form:

$$\chi = \sum_{n=0}^{\infty} \alpha^n \chi_n.$$

where $\alpha = f(P_{\xi}, P, r_\nu)$ is the definite parameter chosen.

The succession of the functions χ_n is determined with the help of the operator:

$$\left.\begin{aligned} & e^{-\zeta^*} A_{R_k}(\chi_n) \\ & = \tfrac{1}{2}[e^{\zeta^*} \textstyle\int e^{-2\zeta^*} R_k(\zeta^*)\, \chi_n(\zeta^*)\, d\zeta^* \\ & - e^{-\zeta^*} \textstyle\int R_k(\zeta^*) \chi_n(\zeta^*)\, d\zeta^*], \end{aligned}\right\} \quad (2.5)$$

so

$$\chi_{n+1} = A_{R_k}(\chi_n); \quad \chi_0 = 1.$$

The proof is given here for the existence of the fundamental solution $e^{-\zeta^*}\mathscr{F}_2(\zeta^*)$.

Let us formulate the main properties of the operator A_{R_k}. Let $|q_\nu| > |P_\nu|$ for coefficients of polynomials of one order

$$Q_m = \sum_{\nu=0}^{m} q_\nu \zeta^\nu \text{ and } P_m = \sum_{\nu=0}^{m} P_\nu \zeta^\nu$$

where q_ν are the coefficients of the polynomial Q_m of one sign. In this meaning we shall use $Q_m \gg P_m$. Besides equation (2.4) we shall consider the following equation:

$$\frac{d^2\chi^*}{d\zeta^{*2}} - \chi^* = \frac{1}{P} P_{\xi} S_k(\zeta^*)\chi^*, \quad (2.6)$$

where

$$S_k(\zeta) = \sum_{\nu=0}^{k} S_\nu \zeta^\nu \gg R_k(\zeta).$$

Then we can prove that $A_{S_k}(Q_m) \gg A_{R_k}(P_m)$.

Thus, if for equation (2.6) we build a formal solution in the form of the succession $\chi_{n+1}^* = A_{S_k}(\chi_n^*)$, and assume that $\chi_0 = \chi_0^* = 1$, then it appears that:

$$\chi_1^* = A_{S_k}(1) \gg A_{R_k}(1) = \chi_1$$

$$\chi_2^* = A_{S_k}(\chi_1^*) \gg A_{R_k}(\chi_1) = \chi_2$$

and

$$\chi_n^* \gg \chi_n.$$

Note that by presenting S_k in the form of $S_k = a(\zeta + N)^k$, and by choosing the constants a and N correspondingly, one may attain the fulfilment of the relation $S_k \gg R_k$. By substituting the variable $2(\zeta + N)\sqrt{(P)} = \kappa$, equation (2.6) is reduced to the form:

$$\frac{d^2\chi^*}{d\kappa^2} - \tfrac{1}{4}\chi^* = \frac{aP_{\xi}}{2^{k+2} P^{(k/2)+1}} \kappa^k \chi^*.$$

By the method of complete induction it was shown [1] that terms of succession of the functions χ_n given in the form:

$$\bar{\chi}_n = \frac{1}{n!(k+1)^n}\left[\kappa + M(k+1) \prod_{m=2}^{n} \frac{m(k+1)}{m(k+1)-1}\right]^{n(k+1)} \quad (2.7)$$

where

$$M \geqslant 2,$$

and of the succession of the solution for equation (2.6) are bound by the condition

$$\bar{\chi}_n \geqslant \chi_n^*. \tag{2.8}$$

Let us prove the convergence of the original $e^{-(\kappa/2)} \sum_{n=0}^{\infty} a^n \bar{\chi}_n$. Apparently, on the basis of equation (2.8) the convergence of the original $e^{-(\kappa/2)} \sum_{n=0}^{\infty} a^n \chi_n^*$ will be proved automatically.

The proof of the convergence is based on the estimation of the expression:

$$\frac{L_{\xi,t}^{-1}(e^{-(\kappa/2)}\, a^{n+1}\, \bar{\chi}_{n+1})}{L_{\xi,t}^{-1}(e^{-(\kappa/2)}\, a^n\, \bar{\chi}_n)}.$$

Taking into account equations (2.1)–(2.3) it may be shown that

$$\left|\frac{L_{\xi,t}^{-1}(e^{-(\kappa/1)}\, a^{n+1}\, \bar{\chi}_{n+1})}{L_{\xi,t}^{-1}(e^{-(\kappa/2)}\, a^n\, \bar{\chi}_n)}\right| \sim a\,\frac{[2(\zeta+N)]^{k+1}}{(n+1)(k+1)} \tag{2.9}$$

Thus, the existence of the original of one of the fundamental solutions for equation (2.4) is proved. Proceeding from equation (2.9) and (2.5) it is easy to show that the fundamental solution for equation (2.4) is $\chi_1 = e^{\zeta^*}\mathscr{F}_1(\zeta^*) \to \infty$ at $\zeta^* \to \infty$ and $\chi_2 = e^{-\zeta^*}\mathscr{F}_2(\zeta^*) \to 0$ at $\zeta^* \to \infty$.

By this it was proved that the solution of equation (1.1) tends to zero at $z \to \infty$ (see equation (1.10)).

Now we shall turn to prove the existence of the original for the image which is the general solution for equation (1.1).

Let us consider the expression [5]:

$$f(P_t) = \frac{1 + \frac{a_1}{P_t^{1/2}} + \frac{a_2}{P_t} + \ldots + \frac{a_n}{P_t^{n/2}} + \ldots}{1 + \frac{b_1}{P_t^{1/2}} + \frac{b_2}{P_t} + \ldots + \frac{b_n}{P_t^{n/2}}} \tag{*}$$

Let us expand the function $f(P_t)$ in a series by $1/P_t^{1/2}$. We then obtain:

$$f(P_t) = \Delta_0 - \Delta_1 \frac{1}{P_t^{1/2}} + \Delta_2 \frac{1}{P_t} + \ldots + (-1)^n \Delta_n \frac{1}{P_t^{n/2}} + \ldots \tag{**}$$

$$\Delta_0 = 1$$

$$\Delta_1 = \begin{vmatrix} 1 & 1 \\ a_1 & b_1 \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} 1 & 1 & 0 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & b_1 \end{vmatrix}$$

$$\Delta_k = \begin{vmatrix} 1 & 1 & 0 & 0 & \ldots & 0 \\ a_1 & b_1 & 1 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{k-1} & b_{k-1} & b_{k-2} & \ldots & & 1 \\ a_k & b_k & b_{k-1} & \ldots & & b_1 \end{vmatrix}$$

The series is convergent outside any ring with a centre in the origin of co-ordinates $P_t = 0$, including all the poles of the function $f(P_t)$. The transformation of it into the language of the variable t may be carried out using the operational relation term by term:

$$\sum_{n=1}^{\infty} \frac{a_n}{P_t^{n/2}} \doteqdot U(t) \sum_{n=1}^{\infty} a_n \frac{t^{n/2}}{P[1+(n/2)]}, \; Re\, P_t > 0$$

This follows from the fact that the sum of any convergent series always equals its Borelevsky's sum:

$$\sum_{n=1}^{\infty} c_n = \int_0^{\infty} e^{-t} \sum_{n=1}^{\infty} \frac{c_n}{(n/2)!}\, t^{n/2}\, dt.$$

Thus, the existence of the original for the image of the type $\chi_1(0)/\chi_2(0)$ is proved (see (1.10)). In an analogous way the existence of the original for any terms in expressions (1.10) and (1.11) may be proved.

Proceeding from the succession of (2.7), it is easy to prove the absolute convergence of derivatives entering equation (1).

Thus, the formal series obtained by our method absolutely converges with all its derivatives, and is in fact the solution of equation (1.1).

REFERENCES

1. O. S. BERLYAND, Ob odnom metode reshenia uravnenia diffuzii (teploprovodnosti), *Inzh.-Fiz. Zh.* N 5 (1959).
2. O. S. BERLYAND and L. A. SOKOLOVSKAYA, O funktsiakh i^n erf $c\,x$, *Inzh.-Fiz. Zh.* N 11 (1958).
3. O. S. BERLYAND, R. I. GAVRILOVA and A. P. PRUDNIKOV, O funktsiakh, udovletvoryaushchikh differentsialnomu uravneniu $y'' \pm 2xy' \pm 2ny = 0$, *Inzh.-Fiz. Zh.* N 9 (1960).
4. O. S. BERLYAND, *Dokl. Akad. Nauk S.S.S.R.* **124**, N 3 (1959).
5. VAN DER POHL and BREMMER, *Operatsionnoe ischislenie na osnove dvukhstoronnego preobrazovaniya Laplasa.* Izd. Inostrannoi Lit., Moscow.

Abstract—A method of solution of the heat conduction (diffusion) equation is developed for a non-stationary three-dimensional problem with convection. A theoretical substantiation of the method is present for the case of a horizontal (longitudinal) velocity component of medium displacement being an arbitrary polynomial relative to the vertical co-ordinate. The solution for the equation is based on the operational method and has the form of a series arranged according to the powers of the generalized Laplace parameter.

Résumé—La méthode de résolution de l'équation de conduction de la chaleur (diffusion) est étudiée dans le cas d'un problème transitoire, tridimensionnel avec convection. Une application théorique de la méthode est présentée dans le cas où la composante de la vitesse horizontale (longitudinale) du déplacement moyen est un polynôme quelconque de la coordonnée verticale. La solution de l'équation est fondée sur la méthode opérationnelle et a la forme d'une série ordonnée suivant les puissances du paramètre généralisé de Laplace.

Zusammenfassung—Die Lösungsmethode der Wärmeleitungsgleichung (Diffusion) wird auf das instationäre dreidimensionale Problem mit Konvektion ausgedehnt. Theoretisch bewiesen wird diese Methode für den Fall, dass die horizontale (longitudinale) Geschwindigkeitskomponente von mittlerer Grösse als willkürliche Polynomfunktion der Vertikal-Koordinate gesetzt werden kann. Die Lösung der Gleichung ist in Form einer Reihe mit steigenden Potenzen des verallgemeinerten Laplace-Parameters angegeben.